MTH 512,Final Exam

Ayman Badawi

QUESTION 1. (Construct new inner products on R^n) Let $n \geq 2$ and $T : R^n \rightarrow R^m$ be a linear transformation (R-homomorphism) that is $one - to - one$, and hence $m \geq n$. Define a new inner product on R^n, denoted by $< , >_N$, such that $< x,\ y >_N = T(x) \cdot T(y)$ for every $x, y \in R^n$, where $\cdot$ is the normal dot product on R^m. Prove that $< , >_N$ is an inner product on R^n, i.e., show that $< , >$ satisfies the axioms of the inner product space. [interesting... since if $n = m$ and T is the identity linear transformation, then $< , >_N$ is just the normal dot product on R^n.]

QUESTION 2. Let $a_1, ..., a_n$ be nonzero real numbers for some $n \geq 2$. Prove that

$$n \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} a_k^{-2}}$$

QUESTION 3. Let A be a symmetric 4×4 matrix such that $1, -1$ are the only eigenvalues of A.

(i) Find all possible rational forms of A

(ii) Evaluate $A^{2023} - A^{2021} + A^{2020} - A^{2018} + 7I_4$.

(iii) Convince me that $A^{-1} = A$.

QUESTION 4. Let A be a 7×7 matrix such that $m_A(\alpha) = (\alpha - 2)^3(\alpha - 5)^2$. Assume $dim(E_2(A)) = 2$.

(i) Find all possible Jordan forms of A.

(ii) For each possible Jordan form, find $dim(E_5(A)), dim(G - E_5(A))$, and $dim(G - E_2(A))$

(iii) For each Jordan form in (i), find the corresponding rational form.

QUESTION 5. (i) Let V and W be inner product vector spaces over R and $T : V \rightarrow W$ be a linear transformation. Then the adjoint operator $T^a : W \rightarrow V$ is a linear transformation. Prove that $Ker(T^a) = Range(T)^{\perp}$. [hint : show $Ker(T^a) \subseteq Range(T)^{\perp}$ and $Range(T)^{\perp} \subseteq Ker(T^a)$. Maybe somewhere you need to notice that if $w \in W$, then $T^a(w) \in V$ and hence $< T(T^a(w)), w >_W = < T^a(w), T^a(w) >_V$]

(ii) Let $< , >$ be the normal dot product on R^n and $T : R^3 \rightarrow R^4$ be a linear transformation such that $Range(T) = span\{(1, 1, 0, 1), (-1, -1, 1, 1)\}$. Find a basis for $Ker(T^a)$. [hint : use (i)]

(iii) Let $< , >$ be the normal dot product on R^k and $T : R^n \rightarrow R^m$ be a linear transformation. Then $F = T\ o\ T^a : R^m \rightarrow R^m$ is a linear transformation. Prove that there are distinct real numbers $a_1, ..., a_k$ such that $m_F(\alpha) = (\alpha - a_1) \cdots (\alpha - a_k)$

QUESTION 6. Let $< , >$ be the normal dot product on R^k and $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$. Then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ is inconsistent. Find the best solution to the system.

QUESTION 7. Given that a matrix A is similar to J, where $J = J_1(3) \oplus J_1(3) \oplus J_4(3) \oplus J_2(4) \oplus J_5(4)$

(i) Find $m_A(\alpha)$.

(ii) Find the rational form of A. [write it as $C(\ ) \oplus \cdots \oplus C(\ )$]

(iii) For each eigenvalue a of A, find $dim(E_a(A))$ and $dim(G - E_a(A))$.

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Question 1: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a L.T. [illegible] $m \geq n$

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let $\langle : \rangle_N : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

s.t. $\langle x, y \rangle_N = T(x) . T(y)$

we show that $\langle , \rangle_N$ is an inner product on $\mathbb{R}^n$

1). $\langle x, x \rangle_N = T(x) . T(x) = T(x)^2 \geq 0 \quad \forall x \in \mathbb{R}^n$

since T is injective $\ker(T) = \{0_{\mathbb{R}^n}\}$

that $T(x) = 0_{\mathbb{R}^m}$ iff $x = 0_{\mathbb{R}^n}$

Hence $\langle x, x \rangle = 0$ iff $x = 0_{\mathbb{R}^n}$

2). $\langle x+y, z \rangle_N = T(x+y) . T(z) = [T(x) + T(y)] . T(z)$

since it is the normal dot product $T(z)$ distributes of $T(x)$ and $T(y)$

$= T(x).T(z) + T(y).T(z) = \langle x, z \rangle + \langle y, z \rangle$

3). $\langle \alpha x, y \rangle = \alpha T(x) . T(y) = \alpha (T(x) . T(y)) = \alpha \langle x, y \rangle$

4). $\langle x, y \rangle = T(x) . T(y) = T(y) . T(x) = \langle y, x \rangle$

using properties of dot product and properties of $\mathbb{R}$

$\langle , \rangle_N$ is an inner product on $\mathbb{R}^n$

[illegible]

Question 2: let $a_1, \dots, a_n$ be non zero real numbers for some $n \geq 2$

Prove that $n \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} a_k^{-2}}$

We know from exa 1 that

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$$

let $b_i = a_i^{-1} \quad \forall 1 \leq i \leq n$

$$\left|\sum_{i=1}^{n} a_i (a^{-1})_i\right| = n \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} (a^{-1})_i^2}$$

$$n \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} a_i^{-2}}$$

Question 3:

let A be a symmetric 4×4 matrix s.t. $1, -1$ are the only eigenvalues

i). Find all possible rational forms of A.

A is symmetric, so A is diagonalizable

$m_A(\alpha) = (\alpha-1)(\alpha+1)$.

$C_A(\alpha) = (\alpha-1)^3(\alpha+1)^1$

$f_3 = m_A(\alpha) = (\alpha-1)(\alpha+1)$

$f_2 = (\alpha-1)$

$f_1 = (\alpha-1)$

$A \approx C((\alpha-1)(\alpha+1)) \oplus C(\alpha-1) \oplus C(\alpha-1)$

$C_A(\alpha) = (\alpha-1)^2(\alpha+1)^2$

$f_2 = m_A(\alpha) = (\alpha-1)(\alpha+1)$

$f_1 = (\alpha-1)(\alpha+1)$

$A \approx C((\alpha-1)(\alpha+1)) \oplus C((\alpha-1)(\alpha+1))$

$C_A(\alpha) = (\alpha-1)^1(\alpha+1)^3$

$f_3 = m_A(\alpha) = (\alpha-1)(\alpha+1)$

$f_2 = (\alpha+1)$

$f_1 = (\alpha+1)$

$A \approx C((\alpha-1)(\alpha+1)) \oplus C(\alpha+1) \oplus C((\alpha+1$

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ii). evaluate $A^{2023} - A^{2021} + A^{2020} - A^{2018} + 7I_4$

let $f(\alpha) = \alpha^{2023} - \alpha^{2021} + \alpha^{2020} - \alpha^{2018} + 7 = \alpha^{2018}(\alpha^5 - \alpha^3 + \alpha^2 - 1) + 7$

$= \alpha^{2018}(\alpha^3(\alpha^2-1) + (\alpha^2-1)) + 7 = \alpha^{2018}((\alpha^2-1)(\alpha^3+1)) + 7$

thus $f(\alpha) = 7 \mod (\alpha^2-1) = 7 \mod m_A(\alpha)$

Hence $f(A) = 7I_4$.

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iii). Convince me that $A^{-1} = A$

A satisfies it's minimal polynomial and $m_A(\alpha)$ is minimal polynomial

s.t. $m_A(A) = A^2 - I_4 = 0$

thus $A^2 = I_4$

but $A^{-1}A = A \cdot A^{-1} = I_4$ hence $A = A^{-1}$

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Question 4: A, 7×7 $\quad m_A(\alpha) = (\alpha-2)^3(\alpha-5)^2$

$\dim(E_2(A)) = 2$

i). if $C_A(\alpha) = (\alpha-2)^3(\alpha-5)^2(\alpha-2)^2$ ①

$A \approx J_2(2) \oplus J_3(2) \oplus J_2(5)$

if $C_A(\alpha) = (\alpha-2)^3(\alpha-5)^2(\alpha-2)^1(\alpha-5)^1$ ②

$A \approx J_1(2) \oplus J_3(2) \oplus J_1(5) \oplus J_2(5)$

ii). for $A \approx J_2(2) \oplus J_3(2) \oplus J_2(5)$

$\dim(E_5(A)) = 1$

$\dim(G\text{-}E_5(A)) = 2$

$\dim(G\text{-}E_2(A)) = 5$

for $A \approx J_1(2) \oplus J_3(2) \oplus J_1(5) \oplus J_2(5)$

$\dim(E_5(A)) = 2$

$\dim(G\text{-}E_5(A)) = 3$

$\dim(G\text{-}E_2(A)) = 4$

iii) for the 1st Jordan form:

$A \approx C((\alpha-2)^3(\alpha-5)^2) \oplus C((\alpha-2)^2)$

for the 2nd Jordan form:

$A \approx C((\alpha-2)^3(\alpha-5)^2) \oplus C((\alpha-2)(\alpha-5))$

Question 5:

i) $T: V \longrightarrow W$

$T^a: W \longrightarrow V$

$\Rightarrow$ let $w \in Ker(T^a) \subseteq W$

then $T^a(w) = 0_V$, let $v \in V$

$\langle T(v), w \rangle_W = \langle v, T^a(w) \rangle_V = 0$

so $w \perp T(v)$ $\forall v \in V$ in particular

Hence $w \perp T(v)$ $\forall v \in V$, w is orthogonal on the basis of Range (T)

this is $w \in$ Range$^{\perp}(T)$ ①

$\Leftarrow$ let $w \in$ Range $(T)^{\perp} \subseteq W$

then $w \perp v$ for every v in Range T.

and $T^a(w) \in V$

so $\langle T(T^a(w)), w \rangle_W = 0 = \langle T^a(w), T^a(w) \rangle_V$

Hence $T^a(w) = 0_V$

so $w \in Ker(T^a)$. ②

from 1 and 2 we get $Ker(T^a) = $ Range $(T)^{\perp}$

Question 5:

ii). Range $(T) = \text{span}\{(1,1,0,1), (-1,-1,1,1)\}$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}_{2\times 4} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Idea OK

$v_1 + v_2 + v_4 = 0 \Rightarrow v_4 = -v_1 - v_2$

$-v_1 - v_2 + v_3 + v_4 = 0 \qquad v_3 = -v_4 + v_1 + v_2 = 2(v_1 + v_2)$

$(\text{Range}(T))^{\perp} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ (these two are linearly independent)

$= \{(1,0,2,-1), (0,1,2,-1)\}$

$\text{Ker}(T^a) = \{(1,0,2,-1), (0,1,2,-1)\}$

iii). $T: \mathbb{R}^n \to \mathbb{R}^m$, $T^a: \mathbb{R}^m \to \mathbb{R}^n$

$F = T \circ T^a : \mathbb{R}^m \to \mathbb{R}^m$

if A is the standard matrix representation of T

then A^T of T^a

$F(v) = A \cdot A^T(v)$

Now AA^T is the standard matrix representation of F

$(AA^T)^T = (A^T)^T A^T = AA^T$ thus it is symmetric, so diagonalizable. Hence $m_x(F)$ is the product of linear factors

i.e. $m_F(x) = (x-a_1)(x-a_2) \cdots (x-a_k)$.

Question 6:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$A^T A X = A^T \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -8 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

the best solution is $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Question 7:

$$A \approx J_1(3) \oplus J_1(3) \oplus J_4(3) \oplus J_2(4) \oplus J_5(4)$$

i). $m_A(\alpha) = (\alpha-3)^4(\alpha-4)^5$

ii). $C_A(\alpha) = (\alpha-3)^4(\alpha-4)^5(\alpha-3)^1(\alpha-3)^1(\alpha-4)^2$

let $f_3 = m_A(\alpha) = (\alpha-3)^4(\alpha-4)^5$

$f_2 = (\alpha-4)^2(\alpha-3)^1$

$f_1 = (\alpha-3)$

$$A \approx C((\alpha-3)^4(\alpha-4)^5) \oplus C((\alpha-4)^2(\alpha-3)^1) \oplus C((\alpha-3)).$$

iii). $\dim(E_3(A)) = 3$

$\dim(G-E_3(A)) = 6$

$\dim(E_4(A)) = 2$

$\dim(G-E_4(A)) = 7$

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